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PHYSICS LETTERS B
Physics Letters B 640 (2006) 196–200
www.elsevier.com/locate/physletb

A note on failure of the ladder approximation to QCD

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Received 10 January 2006; received in revised form 12 March 2006; accepted 21 July 2006

Available online 8 August 2006

Editor: J.-P. Blaizot

Abstract

In this Letter we show that the claim made in [V. Gogohia, Phys. Lett. B 611 (2005) 129] that the ladder approximation to QCD is internally inconsistent is incorrect. The incorrect conclusion in [V. Gogohia, Phys. Lett. B 611 (2005) 129] is based on the incorrect use of a QED-type Ward–Takahashi relation, which does not hold in the ladder approximation to QCD. We give a proof for this fact.

PACS: 11.15.Tk; 12.38.Lg

Keywords: Ladder approximation to QCD

It is well-known that the full dynamical information on any quantum field theory such as quantum chromodynamics (QCD) is contained in the corresponding quantum equations of motion, the so-called Dyson–Schwinger (DS) equations. Due to the fact that the DS equations are an infinite tower of coupled integral equations that relate all of the dressed n-point functions of a quantum field theory to each other, there is no hope for an exact solutions. Thus, in phenomenological applications, one often proceed by making certain simplifications and truncations regarding some subset of n-point functions such that the DS equations are reduced to a closed system of equations which may be solved directly. The simplest of such truncation schemes is the rainbow-ladder approximation. Over the past few years, considerable progress has been made in the framework of the rainbow-ladder approximation of the DS approach [1–5], which provides a successful description of various nonperturbative aspects of strong interaction physics [6–20].

Recently, the ladder approximation is examined by Ref. [21] and the author claims that the ladder approximation to QCD is internally inconsistent. Since the ladder approximation is such a widely used approximation, its internal consistency is an important issue and deserves careful investigation. We have reexamined the reasoning in [21] and found that this claim is incorrect. The incorrect conclusion in [21] is based on the incorrect use of a QED-type Ward–Takahashi relation, which does not hold in the ladder approximation to QCD. In the following we shall first briefly recall the arguments in [21], and then prove that the QED-type Ward–Takahashi relation used there does not hold in the ladder approximation to QCD.

Following the notations in [21], the quark DSE under the ladder approximation reads

$$S^{-1}(p) = S_0^{-1}(p) + i\Sigma(p) = S_0^{-1}(p) + g^2 C_F \int \frac{d^4 l}{(2\pi)^4} \gamma_\alpha S(l) \gamma_\beta D_{\alpha\beta}(p-l), \tag{1}$$

where $i\Sigma(p)$ is the quark self-energy and C_F is the eigenvalue of the quadratic Casimir operator in the fundamental representation ($C_F = (N^2-1)/2N = 4/3$, $N = 3$ for QCD). Here we note that there is a typing error in Eq. (2.1) in Ref. [21] (the sign before

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doi:10.1016/j.physletb.2006.07.065

$g^2 C_F$ should be positive). $S_0^{-1}(p)$ is the free quark propagator

$$S_0^{-1}(p) = -i(\not p - m_0) \tag{2}$$

with m_0 being the current quark mass, and $D_{\alpha\beta}(q)$ is the gluon propagator in an arbitrary covariant gauge

$$D_{\alpha\beta}(q) = -i\left\{\left[g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right]d\left(q^2,\xi\right) + \xi\frac{q_\alpha q_\beta}{q^2}\right\}\frac{1}{q^2}, \tag{3}$$

where ξ is the gauge-fixing parameter. Differentiating both sides of Eq. (1) with respect to p^μ, one gets the differential form of the quark DSE used in [21]

$$\partial_\mu S^{-1}(p) = -i\gamma_\mu + \partial_\mu i\Sigma(p) = -i\gamma_\mu + g^2 C_F \int \frac{d^4 l}{(2\pi)^4}\gamma_\alpha\left[\partial_\mu S(l)\right]\gamma_\beta D_{\alpha\beta}(p-l). \tag{4}$$

The quark–gluon vertex DSE under the ladder approximation reads

$$\Gamma_\mu^a(p,k) = -i\gamma_\mu T^a - g^2\int\frac{d^4 l}{(2\pi)^4}\gamma_\alpha T^b S(l)\Gamma_\mu^a(l,k)S(l-k)\gamma_\beta T^b D_{\alpha\beta}(p-l), \tag{5}$$

where a, b are color indices with T^a for the standard Gell-Mann $SU(3)$ representation and k is the momentum transfer. Assuming analyticity of the vertex at zero momentum transfer, one obtains from the above equation

$$\Gamma_\mu^a(p,0) = -i\gamma_\mu T^a - g^2\int\frac{d^4 l}{(2\pi)^4}\gamma_\alpha T^b S(l)\Gamma_\mu^a(l,0)S(l)\gamma_\beta T^b D_{\alpha\beta}(p-l). \tag{6}$$

At this point the author in Ref. [21] argues that in the ladder approximation one should omit the ghost–quark scattering kernel contained in the Slavnov–Taylor identity of QCD and therefore the Slavnov–Taylor identity reduces to the following QED-type Ward–Takahashi relation

$$k_\mu\Gamma_\mu^a(p,k) = T^a S^{-1}(p) - T^a S^{-1}(p-k). \tag{7}$$

From this relation one derives the following Ward–Takahashi identity relating $\Gamma_\mu^a(p,0)$ and $S^{-1}(p)$

$$\Gamma_\mu^a(p,0) = T^a\partial_\mu S^{-1}(p). \tag{8}$$

Here we note that the above Ward–Takahashi identity plays a key role in Ref. [21]'s proof of the "internal inconsistency" of the ladder approximation to QCD. As we will show below, this Ward–Takahashi identity should not be used because it does not hold in the ladder approximation to QCD, and thus the proof in Ref. [21] is incorrect.

Using the above Ward–Takahashi identity, one can cast Eq. (6) into the form

$$\partial_\mu S^{-1}(p)T^a = -i\gamma_\mu T^a + g^2 T^b T^a T^b\int\frac{d^4 l}{(2\pi)^4}\gamma_\alpha\left[\partial_\mu S(l)\right]\gamma_\beta D_{\alpha\beta}(p-l), \tag{9}$$

where we have used the identity $\partial_\mu S^{-1}(p) = -S^{-1}(p)[\partial_\mu S(p)]S^{-1}(p)$ in deriving the above equation. Making use of the identity $T^b T^a T^b = (C_F - \frac{1}{2}C_A)T^a$ to cancel the $SU(3)$ generator T^a from both sides of the above equation, where C_A is the eigenvalue of the quadratic Casimir operator in the adjoint representation ($C_A = N = 3$ for QCD), one obtains

$$\partial_\mu S^{-1}(p) = -i\gamma_\mu + g^2\left(C_F - \frac{1}{2}C_A\right)\int\frac{d^4 l}{(2\pi)^4}\gamma_\alpha\left[\partial_\mu S(l)\right]\gamma_\beta D_{\alpha\beta}(p-l). \tag{10}$$

Comparing this with Eq. (4) gives

$$-\frac{1}{2}g^2 C_A\int\frac{d^4 l}{(2\pi)^4}\gamma_\alpha\left[\partial_\mu S(l)\right]\gamma_\beta D_{\alpha\beta}(p-l) = 0. \tag{11}$$

From this result and Eq. (4), one concludes that

$$\partial_\mu i\Sigma(p) = 0. \tag{12}$$

The above constraint has only a trivial solution

$$\Sigma(p) = m_c, \tag{13}$$

where m_c is the constant of the dimensions of mass (constant of integration).

According to the above observation the author of Ref. [21] concludes that in the ladder approximation to QCD the quark propagator is a free one, apart from a redefinition of the quark mass, i.e. there is no running/dressed quark mass. Based on this result, the author further argues that the quark propagator in the ladder approximation in fact equals the zero-mass free propagator

$$S(p)=\frac{i}{\not p}, \tag{14}$$

i.e. there is no current quark mass, and the vertex at zero momentum transfer is always trivial

$$\Gamma^a_\mu(p,0)=-i\gamma_\mu T^a. \tag{15}$$

From this the author of Ref. [21] concludes that the ladder approximation to QCD is internally inconsistent, and all the results based on the nontrivial (analytical or numerical) solutions to the quark DSE in the ladder approximation should be reconsidered, and its use in the whole energy/momentum range should be abandoned.

As was noted earlier, the key point of the above arguments is the use of the Ward–Takahashi identity $\Gamma^a_\mu(p,0)=T^a\partial_\mu S^{-1}(p)$, which provides a constraint between the quark–gluon vertex at zero momentum transfer and the inverse quark propagator. Here we want to stress that this relation cannot be used because it does not hold in the ladder approximation to QCD. The following is our proof.

Our starting point is the quark–gluon vertex DSE (5). First let us contract both sides of Eq. (5) with k_μ and obtain

$$k_\mu\Gamma^a_\mu(p,k)=-i\not k T^a-g^2\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha T^b S(l)k_\mu\Gamma^a_\mu(l,k)S(l-k)\gamma_\beta T^b D_{\alpha\beta}(p-l). \tag{16}$$

Then, if the Ward–Takahashi relation (7) holds, we can substitute $k_\mu\Gamma^a_\mu(l,k)$ with $T^aS^{-1}(l)-T^aS^{-1}(l-k)$ and the right-hand side (RHS) of the above equation becomes

$$\begin{aligned}
&-i\not k T^a-g^2\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha T^b S(l)\big(T^aS^{-1}(l)-T^aS^{-1}(l-k)\big)S(l-k)\gamma_\beta T^b D_{\alpha\beta}(p-l)\\
&\quad=-i\not k T^a-g^2T^bT^aT^b\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha\big(S(l-k)-S(l)\big)\gamma_\beta D_{\alpha\beta}(p-l)\\
&\quad=-i\not k T^a+g^2\left(\frac{1}{2}C_A-C_F\right)T^a\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha\big(S(l-k)-S(l)\big)\gamma_\beta D_{\alpha\beta}(p-l)\\
&\quad=T^aS_0^{-1}(p)-T^aS_0^{-1}(p-k)+T^ai\Sigma(p)-T^ai\Sigma(p-k)+\frac{1}{2}g^2C_AT^a\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha\big(S(l-k)-S(l)\big)\gamma_\beta D_{\alpha\beta}(p-l)\\
&\quad=T^aS^{-1}(p)-T^aS^{-1}(p-k)+\frac{1}{2}g^2C_AT^a\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha\big(S(l-k)-S(l)\big)\gamma_\beta D_{\alpha\beta}(p-l),
\end{aligned} \tag{17}$$

where we have made use of the quark DSE (1). One sees that there appears an extra term

$$\frac{1}{2}g^2C_AT^a\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha\big(S(l-k)-S(l)\big)\gamma_\beta D_{\alpha\beta}(p-l) \tag{18}$$

on the RHS of (17). If this extra term does not vanish, the RHS of (17) cannot be equal to $T^aS^{-1}(p)-T^aS^{-1}(p-k)$ and we have a contradiction, showing that the Ward–Takahashi relation (7) cannot hold in the ladder approximation to QCD. This assertion can be proved in different ways. One way is to use the weak coupling expansion. In the weak coupling limit, the quark propagator and the gluon propagator can be expanded in powers of the coupling constant g, with the zeroth order term being the free quark and gluon propagator. Substituting this expansion into (18) we get a power series expansion for the extra term. In order to prove that the extra term (18) does not vanish, it is sufficient to prove the leading term of the expansion of (18) does not vanish. It is apparent that the leading term is obtained by substituting the quark and gluon propagators in (18) by the free ones:

$$S(l-k)\to\frac{i}{\not l-\not k-m_0},\qquad S(l)\to\frac{i}{\not l-m_0},\qquad D_{\alpha\beta}(p-l)\to\frac{-ig_{\alpha\beta}}{(p-l)^2}, \tag{19}$$

where for simplicity we have chosen the Feynman gauge for the gluon propagator. Without loosing generality we may further assume that the bare quark mass m_0 is zero and the leading term of (18) reads

$$\frac{1}{2}g^2C_AT^a\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha\left(\frac{1}{\not l-\not k}-\frac{1}{\not l}\right)\gamma_\alpha\frac{1}{(p-l)^2}. \tag{20}$$

This integral is ultraviolet divergent and can be computed using, for instance, dimensional regularization. In n dimensions, (20) becomes

$$\frac{1}{2}g^2\big(\mu^2\big)^{2-\frac{n}{2}}C_AT^a\int\frac{d^nl}{(2\pi)^n}\gamma_\alpha\left(\frac{1}{\not l-\not k}-\frac{1}{\not l}\right)\gamma_\alpha\frac{1}{(p-l)^2}, \tag{21}$$

where μ is an arbitrary mass scale introduced to make the coupling constant dimensionless. The remaining computation is standard and we only give the results:

$$\frac{1}{2}g^2C_AT^a(2-n)\frac{i(-\pi)^{\frac{n}{2}}}{(2\pi)^n}\frac{\Gamma(2-\frac{n}{2})\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}-1)}{\Gamma(n-1)}\left(\left(\frac{\mu^2}{(p-k)^2}\right)^{2-\frac{n}{2}}(\not p-\not k)-\left(\frac{\mu^2}{p^2}\right)^{2-\frac{n}{2}}\not p\right). \tag{22}$$

This is nonzero and therefore (at least in the weak couping limit) our assertion is proved. The above assertion can also be proved directly by numerical calculations. More specifically, one chooses a suitable model gluon propagator in a specific gauge and use it as input to numerically solve the quark DSE (Eq. (1)) (more details can be found in Refs. [1–5]). Substituting the obtained quark propagator and model gluon propagator into (18), one finds that the extra term does not vanish. Here we remark that from the quark DSE (Eq. (1)) and quark–gluon vertex DSE (Eq. (5)) in the ladder approximation one can obtain nontrivial solutions to the quark DSE, as was demonstrated in the existing literatures [1–20]. From the above results we conclude that the Ward–Takahashi relation (7) cannot hold in the ladder approximation to QCD. At this point the readers can see clearly that if one imposes the Ward–Takahashi relation (7), as was done in Ref. [21], then the extra term (18) must vanish

$$\frac{1}{2}g^2C_AT^a\int\frac{d^4l}{(2\pi)^4}\gamma_\alpha\big(S(l-k)-S(l)\big)\gamma_\beta D_{\alpha\beta}(p-l)\equiv 0. \tag{23}$$

Expanding the $S(l-k)$ in powers of k_μ, one sees that the term linear in k_μ gives exactly Eq. (11), which is just the equation used in Ref. [21] to reach its conclusion (see Eq. (5.4) in that reference).

Here it should be noted that in QCD the quark–gluon vertex function satisfies the Slavnov–Taylor identity which involves the ghost–quark scattering kernel [22] and the author in Ref. [21] assumes in the ladder approximation the above “reduced” Ward–Takahashi identity (7) holds. Since this identity does not hold, just as we have proved above, one cannot make use of it to prove the inconsistency of the ladder approximation to QCD, and the conclusion in Ref. [21] “all the results based on the nontrivial (analytical or numerical) solutions to the quark DSE in the ladder approximation should be reconsidered, and its use in the whole energy/momentum range should be abandoned” is incorrect. Here we also note that in the ladder approximation to QCD one does not have a Ward–Takahashi identity for the quark–gluon vertex (due to $SU(3)$ color structures), but it can be shown that for the color singlet vector $q\bar{q}$ bound state vertex the corresponding Ward–Takahashi identity is valid [7].

Acknowledgements

This work was supported in part by the National Natural Science Foundation of China (under Grant Nos. 10575050, 10135030, 10475057) and the Research Fund for the Doctoral Program of Higher Education (under Grant No. 20030284009).

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